Supersymmetry and Models for Two Kinds of Interacting Particles

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We show that Calogero–Sutherland models for interacting particles have a natural supersymmetric extension. For the construction, we use Jacobians which appear in certain superspaces. Some of the resulting Hamiltonians have a direct physics interpretation as models for two kinds of interacting particles. One model may serve to describe interacting electrons in a lower and upper band of a quasi–one–dimensional semiconductor, another model corresponds to two kinds of particles confined to two different spatial directions with an interaction involving tensor forces.

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Calogero [1, 2] and Sutherland [3] introduced models for interacting particles, which have much in common with a group theoretical construction by Dyson [4]. Various modifications of these models have been studied, see the detailed reviews in Refs. [5, 6] and references therein. A typical model of the Calogero–Sutherland type is defined by the Schrödinger equation for N particles in one dimension at positions $x_n,\ n = 1, \ldots, N$,

$$H\Psi_N^{(\beta)}(x,\kappa) = \left(\sum_{n=1}^{N}\kappa_n^2\right)\Psi_N^{(\beta)}(x,\kappa)\ , \tag{1}$$

with the Hamiltonian

$$H = -\sum_{n=1}^{N}\frac{\partial^2}{\partial x_n^2} + \beta\left(\frac{\beta}{2}-1\right)\sum_{n<m}\frac{1}{(x_n-x_m)^2}\ . \tag{2}$$

The many particle wavefunction $\Psi_N^{(\beta)}(x,\kappa)$ depends on N quantum numbers $\kappa_n,\ n = 1, \ldots, N$ whose squares add up to the energy on the right hand side of Eq. (1). The Hamiltonian (2) consists of a kinetic and a distance dependent interaction term. The strength is measured by the parameter β, the interaction vanishes for $\beta = 0$ and $\beta = 2$. Usually, one adds N confining potentials to the Hamiltonian (2). This renders the system a bound state problem. Apart from this, they do not significantly affect the structure of the model. Thus, we will not work with confining potentials in the sequel.

In this contribution, we present a most natural extension of these Calogero–Sutherland models. In our construction, we employ superspaces. We will arrive at various new models which are likely to be exactly solvable. Calogero–Sutherland models were related to supersymmetric quantum mechanics in Refs. [7, 8]. Our approach and the ensuing models are different from this. Our models also extend a recent supersymmetric construction [9, 10]. Importantly, our models allow for a physics interpretation which is also most natural. One of the models describes a quasi–one–dimensional problem, the other one a two–dimensional one. One has previously tried to generalize the models of the type (1) to higher space dimensions [11, 12]. Again, our construction and the results are different from this.

To prepare for the derivation of the new models in superspace, we briefly sketch the group theoretical connection for the models in ordinary space. With the Vandermonde determinant $\Delta_N(x) = \prod_{n<m}(x_n - x_m)$, the Jacobian on the space of the real–symmetric $N \times N$, the Hermitean $N \times N$ and the quaternion self–dual $2N \times 2N$ matrices labeled with the parameter $\beta = 1, 2, 4$, respectively, takes the form $|\Delta_N(x)|^\beta$. These spaces are non–compact forms of the symmetric spaces $\mathrm{U}(N)/\mathrm{O}(N)$, $\mathrm{U}(N)/1$ and $\mathrm{U}(2N)/\mathrm{Sp}(2N)$. The corresponding radial Laplace–Beltrami operator reads

$$\Delta_x = \sum_{n=1}^{N}\frac{1}{|\Delta_N(x)|^\beta}\frac{\partial}{\partial x_n}|\Delta_N(x)|^\beta\frac{\partial}{\partial x_n}\ . \tag{3}$$

The equation $\Delta_x\Phi_N^{(\beta)}(x,\kappa) = -\left(\sum_{n=1}^{N}\kappa_n^2\right)\Phi_N^{(\beta)}(x,\kappa)$ for the eigenfunction $\Phi_N^{(\beta)}(x,\kappa)$ is mapped onto the N particle Schrödinger equation (1) with the ansatz $\Phi_N^{(\beta)}(x,k) = \Psi_N^{(\beta)}(x,k)/\Delta_N^{\beta/2}(x)\Delta_N^{\beta/2}(k)$. The parameter β is now viewed as positive and continuous, such that the coordinates $x_n,\ n = 1, \ldots, N$ span a space more general than the (radial parts of) the symmetric spaces.

For the supersymmetric generalization, we consider the two sets of k_1 variables $s_{p1},\ p = 1, \ldots, k_1$ and of k_2 variables $s_{p2},\ p = 1, \ldots, k_2$ and the function

$$B_{k_1k_2}(s) = \frac{\prod_{p<q}(s_{p1}-s_{q1})^{\beta_1}\prod_{p<q}(s_{p2}-s_{q2})^{\beta_2}}{\prod_{p,q}(s_{p1}-cs_{q2})^{\sqrt{\beta_1\beta_2}}}\ , \tag{4}$$

with the parameters $\beta_1, \beta_2 \geq 0$ and $c = \pm i$. For certain values of these parameters, Eq. (4) is the Jacobian (or Berezinian) [13] on symmetric superspaces. In the case $\beta_1 = \beta_2 = 2$ and $c = +i$, the function (4) is the Jacobian on Hermitean supermatrices, i.e. on the non–compact form of $\mathrm{U}(k_1/k_2)/1$ [14, 15], where $\mathrm{U}(k_1/k_2)$ is

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the unitary supergroup [13, 16, 17]. Apart from an absolute value sign which is unimportant here, the choices $\beta_1 = 1,\ \beta_2 = 4,\ c = +i$ and $\beta_1 = 4,\ \beta_2 = 1,\ c = -i$ in Eq. (4) yield the Jacobians [14, 18] for the two forms of the symmetric superspace $\mathrm{Gl}(k_1/2k_2)/\mathrm{OSp}(k_1/2k_2)$, respectively. They are denoted AI|AII and AII|AI in the classification of Ref. [19]. Here, $\mathrm{Gl}(k_1/2k_2)$ is the general linear supergroup and $\mathrm{OSp}(k_1/2k_2)$ is the orthosymplectic supergroup [13, 16, 17]. The imaginary unit in the parameter c stems from a Wick–type–of rotation of the variables s_{p2}, which was performed for a convergence reason [14]. Although not needed here, we keep it for now to make the notation compatible with the literature. The function (4) induces the operator

$$\Delta_s = \frac{1}{\sqrt{\beta_1}}\sum_{p=1}^{k_1}\frac{1}{B_{k_1k_2}(s)}\frac{\partial}{\partial s_{p1}}B_{k_1k_2}(s)\frac{\partial}{\partial s_{p1}} + \frac{1}{\sqrt{\beta_2}}\sum_{p=1}^{k_2}\frac{1}{B_{k_1k_2}(s)}\frac{\partial}{\partial s_{p2}}B_{k_1k_2}(s)\frac{\partial}{\partial s_{p2}} \ . \tag{5}$$

The prefactors $1/\sqrt{\beta_1}$ and $1/\sqrt{\beta_2}$ in front of the sums are such that Δ_s becomes the radial Laplace–Beltrami operator on the three symmetric superspaces mentioned above for the corresponding choices of the parameters β_1, β_2 and c. However, we emphasize that arbitrary positive values for β_1 and β_2 will be considered in the sequel, while the parameter c remains restricted to $c = \pm i$. We map the eigenvalue equation for the operator Δ_s with eigenfunctions $\varphi_{k_1k_2}^{(c,\beta_1,\beta_2)}(s,r)$ onto the equation

$$\widetilde{H}\psi_{k_1k_2}^{(c,\beta_1,\beta_2)}(s,r) = \left(\sum_{p=1}^{k_1}\frac{r_{p1}^2}{\sqrt{\beta_1}} + \sum_{p=1}^{k_2}\frac{r_{p2}^2}{\sqrt{\beta_2}}\right)\psi_{k_1k_2}^{(c,\beta_1,\beta_2)}(s,r) \ , \tag{6}$$

where the wavefunctions are related by the ansatz $\varphi_{k_1k_2}^{(c,\beta_1,\beta_2)}(s,r) = \psi_{k_1k_2}^{(c,\beta_1,\beta_2)}(s,r)/(B_{k_1k_2}(s)B_{k_1k_2}(r))^{1/2}$. The resulting operator reads

$$\widetilde{H} = -\frac{1}{\sqrt{\beta_1}}\sum_{p=1}^{k_1}\frac{\partial^2}{\partial s_{p1}^2} - \frac{1}{\sqrt{\beta_2}}\sum_{p=1}^{k_2}\frac{\partial^2}{\partial s_{p2}^2} + \sum_{p<q}\frac{g_{11}}{(s_{p1}-s_{q1})^2} + \sum_{p<q}\frac{g_{22}}{(s_{p2}-s_{q2})^2} - \sum_{p,q}\frac{g_{12}}{(s_{p1}-cs_{q2})^2} \tag{7}$$

with constants

$$g_{jj} = \sqrt{\beta_j}\left(\frac{\beta_j}{2}-1\right) \ , \quad j=1,2 \ ,$$
$$g_{12} = \frac{1}{2}\left(\sqrt{\beta_1}-\sqrt{\beta_2}\right)\left(\frac{1}{2}\sqrt{\beta_1\beta_2}+1\right) \ . \tag{8}$$

The constant on the right hand side of Eq. (6) is interpreted as energy later on. We write it in terms of two sets of variables $r_{p1},\ p = 1,\ldots,k_1$ and $r_{p2},\ p = 1,\ldots,k_2$ to which we refer as quantum numbers. It turns out convenient to take the scaling factors $1/\sqrt{\beta_1}$ and $1/\sqrt{\beta_2}$ into the definition. In Ref. [9], a similar construction was performed. However, the resulting operator depends on one parameter only. The natural dependence on two strength parameters β_1 and β_2 is an essential point in the present study. Thus, our approach contains the one in Ref. [9] as a special case.

The family of models (7) ought to be exactly solvable. For the parameter values corresponding to the symmetric superspaces, the wavefunctions can be written as supergroup integrals [15, 20, 21]. Thus, the wavefunctions are uniquely characterized by the two sets of quantum numbers $r_{p1},\ p = 1,\ldots,k_1$ and $r_{p2},\ p = 1,\ldots,k_2$. This feature should extend to all values $\beta_1, \beta_2 \geq 0$, since the operators Δ_s and $\widetilde{H}$ are analytic in the upper right quadrant of the complex plane $\beta_1 + i\beta_2$. However, a final statement needs more mathematical work.

We present a physical interpretation. The operators Δ_s and $\widetilde{H}$ are not Hermitean, which is solely due to the Wick–type–of rotation mentioned above. We consider $c = +i$ and undo this rotation by replacing is_{p2} with s_{p2}. By also introducing the momenta $\pi_{p1} = -i\partial/\partial s_{p1}$ and $\pi_{p2} = -i\partial/\partial s_{p2}$, we transform the operator (7) into the Hermitean Hamiltonian for two kinds of k_1 particles at positions $s_{p1},\ p = 1,\ldots,k_1$ and k_2 particles at positions $s_{p2},\ p = 1,\ldots,k_2$ on the s axis,

$$H = \sum_{p=1}^{k_1}\frac{\pi_{p1}^2}{2m_1} + \sum_{p=1}^{k_2}\frac{\pi_{p2}^2}{2m_2} + \sum_{p<q}\frac{g_{11}}{(s_{p1}-s_{q1})^2} - \sum_{p<q}\frac{g_{22}}{(s_{p2}-s_{q2})^2} - \sum_{p,q}\frac{g_{12}}{(s_{p1}-s_{q2})^2} \ , \tag{9}$$

with now canonical conjugate variables satisfying $[\pi_{pj}, s_{ql}] = \delta_{pq}\delta_{jl}$. We notice that the mass $m_1 = \sqrt{\beta_1/4}$ is positive, while the mass $m_2 = -\sqrt{\beta_2/4}$ is negative. The interactions are according to Eq. (8) repulsive or attractive, depending on the choices for β_1 and β_2. The particles of the same kind interact, this interaction vanishes for $\beta_1 = 2$ or $\beta_2 = 2$. Two particles of different kind also interact. This interaction vanishes for $\beta_1 = \beta_2$ and the problem decouples into two known models (2). For $k_1 = 0$ or $k_2 = 0$, we recover the models (2).

The Hamiltonian (9) may also serve to model the motion of electrons in a quasi–one–dimensional semiconductor, see Fig. 1. The particles at positions s_{p1} with positive mass m_1 are identified with the electrons subject to a periodic potential in the upper band close to the gap. The electrons in the lower band have a dispersion relation ϵ_k as function of the wave number k whose second derivative, i.e. the inverse mass, is negative [22]. They are identified with the particles at positions s_{p2} that have negative mass m_2.

The models (2) are based on the ordinary unitary group and on associated symmetric spaces. There are other models in ordinary space related to the ordinary

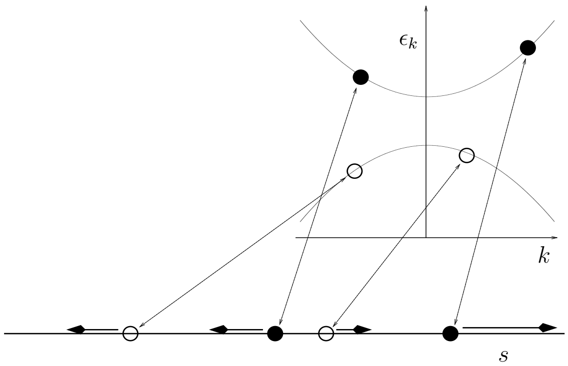

FIG. 1: Electrons in the upper (black circles) and lower (open circles) band of a quasi–one–dimensional semiconductor. The dispersion relations ϵ_k as function of the wave number k are indicated by the parabola and the inverted parabola. The particles are then mapped onto the s axis in the bottom part.

orthogonal and symplectic groups [5]. They have a natural supersymmetric extension as well. We introduce the two sets of $2k_1$ variables $s_{p1},\ p=1,\ldots,k_1$ and of $2k_2$ variables $s_{p2},\ p=1,\ldots,2k_2$. We notice that the number of variables in the second set has to be even. Instead of $2k_1$, we could also consider an odd number $2k_1+1$. As the ensuing models differ only slightly, we restrict ourselves to $2k_1$. We study the function

$$C_{2k_1 2k_2}(s) = \frac{\prod_{p<q}(s_{p1}^2-s_{q1}^2)^{\beta_1}\prod_{p<q}(s_{p2}^2-s_{q2}^2)^{\beta_2}\prod_{p=1}^{k_2}s_{p2}^{\beta_2}}{\prod_{p,q}(s_{p1}^2+s_{q2}^2)^{\sqrt{\beta_1\beta_2}}}\ , \quad (10)$$

which is the Jacobian (or Berezinian) on the supergroup $\mathrm{OSp}(2k_1/2k_2)$ for $\beta_1=\beta_2=2$, see a derivation in Ref. [23]. For $\beta_1=1$, $\beta_2=4$ and $\beta_1=4$, $\beta_2=1$, Eq. (10) gives the Jacobians on the two forms of the symmetric superspace $\mathrm{OSp}(2k_1/2k_2)/\mathrm{Gl}(k_1/k_2)$ which are named CI|DIII and DIII|CI in Ref. [19]. We proceed exactly as before and derive an eigenvalue equation of the form (6) where the operator now reads

$$\begin{aligned}\widetilde{H} \ =\ & -\frac{1}{\sqrt{\beta_1}}\sum_{p=1}^{k_1}\frac{\partial^2}{\partial s_{p1}^2}-\frac{1}{\sqrt{\beta_2}}\sum_{p=1}^{k_2}\frac{\partial^2}{\partial s_{p2}^2}\\ & -g_{11}\sum_{p<q}\frac{2s_{p1}^2+2s_{q1}^2}{\left(s_{p1}^2-s_{q1}^2\right)^2}\\ & +g_{22}\left(\sum_{p<q}\frac{2s_{p2}^2+2s_{q2}^2}{\left(s_{p2}^2-s_{q2}^2\right)^2}+\sum_{p=1}^{k_2}\frac{1}{2s_{p2}^2}\right)\\ & -g_{12}\sum_{p,q}\frac{2s_{p1}^2-2s_{q2}^2}{\left(s_{p1}^2+s_{q2}^2\right)^2}+\sum_{p,q}\frac{2h_{12}}{s_{p1}^2+s_{q2}^2}\ ,\end{aligned} \quad (11)$$

with g_{11}, g_{22} and g_{12} as given in Eq. (8) and with

$$h_{12}=\frac{1}{4}\sqrt{\beta_1\beta_2}\left(\sqrt{\beta_1}-\sqrt{\beta_2}\right)\ . \quad (12)$$

The operator (11) remains invariant when replacing any of the variables by its negative. Due to this symmetry, $\widetilde{H}$ itself is an Hermitean operator and can be viewed as a Hamiltonian, there is no need to undo the Wick–type–of rotation. We now come to a physical interpretation. In a straightforward calculation, the Hamiltonian $H=2\widetilde{H}$ can be cast into the form

$$\begin{aligned}H \ =\ & \sum_{p=1}^{2k_1}\frac{\pi_{p1}^2}{2m_1}+\sum_{p=1}^{2k_2}\frac{\pi_{p2}^2}{2m_2}+\sum_{p=1}^{2k_1}\frac{f_1}{s_{p1}^2}+\sum_{p=1}^{2k_2}\frac{f_2}{s_{p2}^2}\\ & +\sum_{p<q}\frac{h_{11}}{\left(s_{p1}-s_{q1}\right)^2}+\sum_{p<q}\frac{h_{22}}{\left(s_{p2}-s_{q2}\right)^2}\\ & -\sum_{p,q}\frac{h_{12}}{s_{p1}^2+s_{q2}^2}\\ & +\sum_{p,q}\frac{(\vec{e}_{pq}\cdot\vec{\sigma}_1)\,(\vec{e}_{pq}\cdot\vec{\sigma}_2)-\vec{\sigma}_1\cdot\vec{\sigma}_2/2}{s_{p1}^2+s_{q2}^2}\ .\end{aligned} \quad (13)$$

We exploit the symmetry by writing H as a Hamiltonian for $2k_1$ plus $2k_2$ particles, i.e. k_1 plus k_2 pairs of particles sitting at positions $(-s_{p1},+s_{p1})$ on the s_1 axis and $(-s_{p2},+s_{p2})$ on the s_2 axis. This is a two–dimensional situation. Here, we assume that the initial condition is invariant under mirror reflection of the positions s_{p1} and s_{p2}. Each particle on the s_j axis carries a (fixed and non–quantized) dipole vector $\vec{\sigma}_j=\sigma(\cos\vartheta_j,\sin\vartheta_j),\ j=1,2$. The masses $m_j=\sqrt{\beta_j/4},\ j=1,2$ are both positive. The interaction comprises three parts. First, there are central forces with strengths

$$f_1=+\frac{\beta_1}{8}\left(\frac{\beta_1}{2}-1\right)\quad\text{and}\quad f_2=-\frac{\beta_2}{8}\left(\frac{\beta_2}{2}-1\right)\ . \quad (14)$$

Second, there are distance dependent forces between the particles on the same axis with strengths

$$h_{jj}=\sqrt{\beta_j}\left(\frac{\beta_j}{2}-1\right)+\sigma^2\cos 2\vartheta_j\ ,\quad j=1,2 \quad (15)$$

and there is a distance dependent force between the particles on different axes with strength h_{12} as given by Eq. (12). Third, there are tensor or dipole–dipole forces. The last term of the Hamiltonian (13) is a two–dimensional dipole–dipole interaction [24] between the particles on different axes. The unit vector $\vec{e}_{pq}$ points from the particle p on the s_1 axis to the particle q on the s_2 axis. The strength of the dipoles follows from the relation $\sigma^2\cos(\vartheta_1+\vartheta_2)=2g_{12}$ with g_{12} as defined in Eq. (8). Consistently, the tensor force also acts between the dipoles on the same axis. As the latter are parallel, the tensor force acquires a form identical to the distance dependent interaction. This explains the additional term in the constants $h_{jj},\ j=1,2$ as compared to the constants g_{jj}. In Fig. 2 we illustrate the model. Other choices of the parameters are also possible, leading, for example, to different angles of the dipoles on the positive and negative side of the same axis, see Fig. 2.

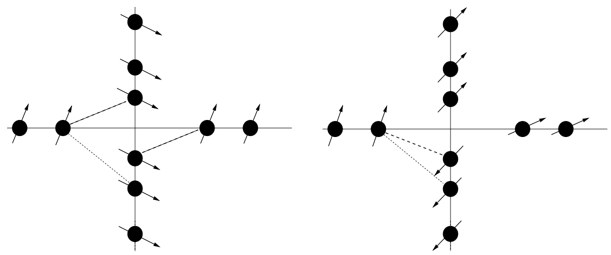

FIG. 2: Two realizations of the two–dimensional model. Left: $2k_1 = 4$ particles on the s_1 axis and $2k_2 = 6$ particles on the s_2 axis. The dipole vectors on the same axis have the same direction. Tensor forces are indicated as thicker and thinner dashed lines, corresponding to the strength of the force. Right: a case with different directions of the dipole vectors on different sides of the same axis.

Again, the family of models (13) should be exactly solvable. For the parameter values corresponding to $\mathrm{OSp}(2k_1/2k_2)$, the wavefunctions are supergroup integrals [23]. Analytical continuation in $\beta_1 + i\beta_2$ should be possible. A rigorous proof has yet to be given.

To gain some first intuition for the solutions of these models, we consider the simplest case $k_1 = k_2 = 1$ and $c = +i$ of Eq. (6). The operator (7) has then a simple structure which yields the wavefunction

$$\psi_{11}^{(+i,\beta_1,\beta_2)}(s,r) = \exp\left(\pm\frac{i\left(\sqrt{\beta_1}s_{11} - i\sqrt{\beta_2}s_{12}\right)(r_{11} - ir_{12})}{\sqrt{\beta_1} - \sqrt{\beta_2}}\right) \frac{z^\nu \mathbf{H}_\nu^\mp(z)}{((s_{11} - is_{12})(r_{11} - ir_{12}))^{\sqrt{\beta_1\beta_2}/2}} \ . \quad (16)$$

Here, $\mathbf{H}_\nu^\mp(z)$ is the Hankel function [25] of order $\nu = \sqrt{\beta_1\beta_2}/2 + 1/2$ and we use the dimensionless, complex variable $z = (\sqrt{\beta_2}r_{11} - i\sqrt{\beta_1}r_{12})(s_{11} - is_{12})/(\sqrt{\beta_2} - \sqrt{\beta_1})$. The appearance of the differences in the denominator in Eq. (16) is typical for superspaces, see Refs. [21]. In ordinary space, those differences are found in the numerator, see Refs. [26, 27]. In particular, this affects the behavior of the wavefunctions at the origin. Work on further analytical results is in progress.

In conclusion, we derived natural supersymmetric extensions of models for interacting particles. The corresponding physics is most natural as well, involving two kinds of particles. We presented two possible applications, a quasi–one–dimensional and a two–dimensional one. From a more general perspective, one might say that our results yield a conceptually new interpretation of supersymmetry. In high–energy physics, the physical bosons and fermions are represented by commuting and anticommuting variables, see Ref. [28]. This is also so in the interacting boson–fermion model [29] of nuclear physics. In chaotic and disordered systems [14], the commuting and anticommuting variables serve to considerably reduce the numbers of the degrees of freedom, they are not seen as physical particles. Here, we showed that the radial coordinates on certain superspaces can be viewed as the positions of interacting particles.

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